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ON ε-REPRESENTATIONS

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ABSTRACT
For certain classes of groups we show that a map to the group of unitary transformations of a Hilbert space which is "almost" a homomorphism is uniformly close to a unitary representation.

V. Milman asked me the following question: Let $\rho : O(n) \to O(N)$ be a map which is "almost" a representation, that is, $|\rho(gg') - \rho(g)\rho(g')|$ is small for all $g, g' \in O(N)$. Is it true that ρ is near to an actual representation of $O(n)$? This paper is a particular answer to this question.

I want to express my gratitude to H. Furstenberg and B. Weiss for their very helpful discussions, and especially to V. Milman for bringing my attention to this problem.

After this paper was written it was called to the author's attention that by similar methods the result of Theorem 1 for compact groups was obtained in [1].

Let U be a topological group and d be a right invariant metric on U which defines the topology of U. Let G be another topological group, R be a continuous map $\rho : G \to U$ and $\varepsilon > 0$ be a real number. We say that ρ is an ε-homomorphism if $d(\rho(gg'), \rho(g)\rho(g')) \leqq \varepsilon/2$ for any $g, g' \in G$. Of course, for any homomorphism $\pi : G \to U$ and a continuous map $\rho : G \to U$ such that $d(\rho(g), \pi(g)) \leqq \varepsilon/4$, ρ is an ε-homomorphism. The following question arises naturally: when is an ε-homomorphism ρ a small perturbation of a homomorphism π?

We start with an example. Let U be the additive group Z_2 of integral dyadic numbers and $d(u, u') = |u - u'|$ where $|\ \ |$ is the usual norm on Z_2.

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PROPOSITION 1. *For any $\varepsilon > 0$ there exists a finite group G and an ε-homomorphism $\rho : G \to U$ such that for any homomorphism $\pi : G \to U$ we have* $\max_{g \in G} d(\rho(g), \pi(g)) = 1$.

PROOF. Take an integer n such that $1/n^2 < \varepsilon$, take $G = Z/2^{n+1}Z$ and define the map $\rho : G \to U$ by $\rho(g) = \tilde{g}$ where $\tilde{g} \in Z \subset Z_2$ is the representative of g such that $0 \leqq g < 2^{n+1}$. It is clear that ρ is an ε-homomorphism.

On the other hand U does not have any torsion. Therefore, the only homomorphism $\pi : G \to U$ is $\pi \equiv 0$. The proposition is proved.

We restrict ourselves to some special special class of groups U with a metric to get some positive results.

Let V be a Banach space. We denote by $L(V)$ the ring of bounded linear operators on V and by $U(V)$ (or simply U) the group of isometries. $L(V)$ has a natural structure of a Banach space and $U(V)$ acts on $L(V)$ by conjugation $\mathrm{Ad}(u)(A) = u^{-1}Au$ for $u \in U$, $A \in L(V)$. We denote by $\| \ \|$ the norms on V and on $L(V)$ and define

$$d(u, u') \stackrel{\text{def}}{=} \|u - u'\| \qquad \text{for } u, u' \in U(V).$$

We will study ε-morphisms $G \to U(V)$ which we call ε-representations.

Let G be a topological group, and $\rho : G \to U$ be an ε-representation.

We denote by C^k_ρ the Banach space of continuous bounded functions $c : G^k \to V$ with the norm

$$|c| = \sup_{g_1, \cdots, g_k \in G} \|c(g_1, \cdots, g_k)\|.$$

We denote by $d^k_\rho : C^k_\rho \to C^{k+1}_\rho$ the linear map given by

$$(d^j_\rho c)(g_1, \cdots, g_{k+1}) = \rho(g_1)c(g_2, \cdots, g_{k+1}) + \sum_{i=1} (-1)^i c(g_1, \cdots, g_i g_{i+1}, \cdots, g_{k+1})$$
$$+ (-1)^{k+1} c(g_1, \cdots, g_k).$$

It is clear that

$$\|d^{k+1}_\rho \circ d^k_\rho\| \leqq \varepsilon$$

We say that an ε-representation ρ is ε-acyclic if for any $c \in C^k_\rho$ there exists $b \in C^{k-1}_\rho$ such that $\|b\| \leqq \|c\|$ and $\|d^{k-1}_\rho b - c\| \leqq \varepsilon \|c\| + \|d^k_\rho c\|$.

Let B_ρ be the Banach space of bounded continuous functions $f : G \to V$ with the norm $\|f\| \stackrel{\text{def}}{=} \sup_{g \in G} \|f(g)\|$.

We denote by $r: G \to U(B_\rho)$ the action of G on B_ρ by right shifts and by $i: V \to B_\rho$ the imbedding given by $i(v)(g) = \rho(g)v$, $v \in V$, $g \in G$. We say that a linear map $I: B_\rho \to V$ is an ε-mean if $\|I\| = 1$, $I \circ i = \mathrm{Id}$ and $\|I \circ r(g) - \rho(g)I\| \leqq \varepsilon$ for any $g \in G$.

LEMMA 1. *If there exists an ε-mean I on B_ρ then ρ is ε-acyclic.*

PROOF. Let $I_k : C_\rho^k \to C_\rho^{k-1}$ be a linear map given by

$$(I_k c)(g_1, \cdots, g_{k-1}) \overset{\text{def}}{=} I(c(g, g_1, \cdots, g_{k-1}))$$

where we consider $c(g, g_1, \cdots, g_{k-1})$ as a V-valued function on G. Then

$$\begin{aligned}
&(d_\rho^{k-1} \circ I_k + I_{k+1} \circ d_\rho^k)(c)(g_1, \cdots, g_k) - c(g_1, \cdots, g_k) \\
&\quad = (\rho(g_1) \circ I)(c(g, g_2, \cdots, g_k)) \\
&\qquad + \sum_{i=1}^{k-1} (-1)^i I(c(g, g_1, \cdots, g_i g_{i+1}, \cdots, g_k)) \\
&\qquad + (-1)^k I(c(g, g_1, \cdots, g_{k-1})) + I(\rho(g)c(g_1, \cdots, g_k)) - I(c(gg_1, g_2, \cdots, g_k)) \\
&\qquad + \sum_{i=2}^{k} (-1)^i I(c(g, \cdots, g_{i-1}g_i, \cdots, g_k)) + (-1)^{k+1} I(c(g, g_1, \cdots, g_{k-1})) \\
&\qquad\qquad - c(g_1, \cdots, g_k) \\
&\quad = (\rho(g_1) \circ I)c(g, g_2, \cdots, g_k) - I(c(gg_1, g_2, \cdots, g_k)) + I(\rho(g)c(g_1, \cdots, g_k)) \\
&\qquad\qquad - c(g_1, \cdots, g_k) \\
&\quad = (\rho(g_1) \circ I - I \circ r(g_1))(c(g, g_2, \cdots, g_k)).
\end{aligned}$$

Therefore $\|d_\rho^{k-1} \circ I_k(c) + I_{k+1} \circ d_\rho^k(c) - c\| \leqq \varepsilon \|c\|$.
Now define $b = I_k c$. Then

$$\|b\| \leqq \|c\| \quad \text{and} \quad \|d_\rho^{k-1} b - c\| \leqq \varepsilon \|c\| + \|d_\rho^k c\|.$$

The lemma is proved.

LEMMA 2. *If an ε-representation ρ satisfies one of the following conditions:*
(a) *G is compact,*
(b) *G is an amenable group and V is a reflexive space,*
(c) *G is an amenable group, V is the Banach space of bounded operators in a Hilbert space H and $\rho = \mathrm{Ad} \circ \tilde{\rho}$ for an ε-representation $\tilde{\rho}$ of G on H,*
then there exists an ε-mean on B_ρ.

PROOF. (a) We define $(If) =^{\text{def}} \int_G \rho(g)^{-1} f(g)dg$ where dg is the Haar measure on G such that $\int_G dg = 1$. It is clear that $\|I\| = 1$ and $I \circ i = \text{Id}$. Therefore, we only have to check that $\|\rho(g) \circ I - I \circ r(g)\| \leqq \varepsilon$ for any $g \in G$. But

$$(I \circ r(g))f = \int_G \rho^{-1}(g')f(g'g)dg' = \int_G \rho^{-1}(g'g^{-1})f(g')dg'.$$

Therefore

$$\|\rho(g) \cdot If - I \circ r(g)f\| \leqq \max_{g,g \in G} \|\rho(g)(g'^{-1}) - \rho^{-1}(g'g^{-1}) \qquad \text{for any } f \in B_\rho.$$

Since ρ is an ε-representation, we have $\|\rho(g) \circ I - I \circ r(g)\| \leqq \varepsilon$.

(b) Since G is an amenable group, there exists a continuous G-invariant linear functional m on $L^\infty(G)$ such that $m(1) = 1$ and $\|m\| = 1$. For any $f \in B_\rho$ we define $I(f)$ as an element in the double dual of V by $I(f)(\lambda) =^{\text{def}} m(a_\lambda(g))$ where $a_\lambda(g) =^{\text{def}} \lambda(\rho(g)^{-1}f(g))$. Since V is a reflexive Banach space, we can consider I as a map from B_ρ to V. Then

$$\|I\| = \max_{f \in B_\rho, \|f\|=1} \|I(f)\| = \max_{\substack{f \in B_\rho, \|f\|=1 \\ \lambda \in V, \|\lambda\|=1}} |\lambda(I(f))|.$$

But $|\lambda(I(f))| = |m(a_\lambda(g))| \leqq \sup_{g \in G} |a_\lambda(g)| \leqq \|\lambda\| \cdot \|f(g)\| \leqq \|\lambda\| \cdot \|f\|$. Therefore $\|I\| \leqq 1$. It is clear that $I \circ i = \text{Id}$ and the proof of the inequality $\|\rho(g)I - I \circ r(g)\| \leqq \varepsilon$ is completely analogous to the one in (a).

(c) For any $f \in B_\rho$ we first define a bilinear form $\tilde{I}(f)$ on H by $\tilde{I}(f)(h_1, h_2) \overset{\text{def}}{=} m(a_{h_1,h_2}(g))$ where $a_{h_1,h_2}(g) \overset{\text{def}}{=} (f(g)\tilde{\rho}(g)^{-1}h_1, \tilde{\rho}(g)^{-1}h_2)$. It is clear that $|\tilde{I}(f)(h_1, h_2)| \leqq \|f\| \cdot \|h_1\| \cdot \|h_2\|$ for any $h_1, h_2 \in H$. Therefore, our bilinear form $\tilde{I}(f)$ defines a bounded linear operator $I(f)$ on H and $\|I(f)\| \leqq \|f\|$. It is now easy to check that $I : B_\rho \to V$ is an ε-mean. Lemma 2 is proved.

THEOREM 1. *Let G be an amenable group and $\tilde{\rho} : G \to U$ be an ε-representation of G into the group U of unitary transformations of a Hilbert space H for $\varepsilon < 1/100$. Then there exists a representation $\pi : G \to U$ such that $\|\tilde{\rho}(g) - \pi(g)\| \leqq \varepsilon$ for all $g \in G$.*

We start with the following result.

PROPOSITION 2. *Suppose that $\tilde{\rho}$ satisfies the conditions of the theorem. Then there exists an ε_1-representation $\tilde{\rho}_1 : G \to U$ for $\varepsilon_1 = 5\varepsilon^2$ such that $\|\tilde{\rho}(g) - \tilde{\rho}_1(g)\| \leqq \varepsilon/2 + \varepsilon^2$.*

PROOF OF THE PROPOSITION. Let V be the space of bounded linear operators on H. We denote by ρ the ε-representation of G on V given by $\rho = \text{Ad} \circ \tilde{\rho}$. For any $g, g' \in G$ we define $W(g, g') \in U$ by $\tilde{\rho}(g)\tilde{\rho}(g') = W(g, g')\tilde{\rho}(gg')$. By the assumption $\|W(g, g') - \mathrm{Id}\| \leqq \varepsilon/2$. We now compute the triple product $\tilde{\rho}(g_1)\tilde{\rho}(g_2)\tilde{\rho}(g_3)$ in two ways.

$$\tilde{\rho}(g_1)\tilde{\rho}(g_2)\tilde{\rho}(g_3) = W(g_1, g_2)\tilde{\rho}(g_1g_2)\tilde{\rho}(g_3) = W(g_1, g_2)W(g_1g_2, g_3)\tilde{\rho}(g_1g_2g_3).$$

On the other hand

$$\begin{aligned}\tilde{\rho}(g_1)\tilde{\rho}(g_2)\tilde{\rho}(g_3) &= \tilde{\rho}(g_1)W(g_2, g_3)\tilde{\rho}(g_2g_3)\\ &= \rho(g_1)(W(g_2, g_3))\tilde{\rho}(g_1)\tilde{\rho}(g_2g_3)\\ &= \rho(g_1)(W(g_2, g_3))W(g_1, g_2g_3)\tilde{\rho}(g_1g_2g_3).\end{aligned}$$

Therefore

$$(*) \qquad W(g_1, g_2)W(g_1g_2, g_3) = \rho(g_1)(W(g_2, g_3))W(g_1g_2g_3).$$

For any $u \in U$ such that $\|u - \mathrm{Id}\| < 1$ we define

$$\ln u \stackrel{\text{def}}{=} \sum_{i=1}^{\infty} (-1)^{i-1} \frac{u^i}{i}.$$

LEMMA 3. *Let $u, u' \in U$ be such that $\|u - \mathrm{Id}\|$, $\|u' - \mathrm{Id}\| \leqq \varepsilon/2$. Then $\|\ln(uu') - (\ln u + \ln u')\| \leqq \varepsilon^2/2$ for $\varepsilon < 1/100$.*

PROOF. Clear.

We define an element α in C_ρ^2 by $\alpha(g, g') = \ln W(g, g')$. It is clear that $\|\alpha\| \leqq \varepsilon/2 + \varepsilon^2/2$. It follows from (*) and Lemma 3 that $\|d_\rho^2\alpha\| \leqq \varepsilon^2$. Therefore by Lemma 2(c) there exists $b \in C_\rho^1$ such that $\|b\| \leqq \varepsilon/2 + \varepsilon^2/2$ and $\|d_\rho^1 b - \alpha\| \leqq 2\varepsilon^2$. It is easy to see that $\alpha(g, g')$ and $b(g)$ are skew Hermitian operators.

We now define the map $\tilde{\rho}_1 : G \to U$ by $\tilde{\rho}_1(g) = \exp(b(g))\rho(g)$. It is clear that $\|\tilde{\rho}(g) - \tilde{\rho}_1(g)\| \leqq \varepsilon/2 + \varepsilon^2$ for any $g \in G$ and

$$\begin{aligned}&\tilde{\rho}_1(gg')^{-1}\tilde{\rho}_1(g)\tilde{\rho}_1(g')\\ &\quad = \tilde{\rho}(gg')^{-1}\exp(-b(gg'))\exp b(g)\tilde{\rho}(g)\exp b(g')\tilde{\rho}(g')\\ &\quad = \tilde{\rho}(g')^{-1}\tilde{\rho}(g)^{-1}W(g, g')\exp(-b(gg'))\exp b(g)\exp(\tilde{\rho}(g)b(g'))\tilde{\rho}(g)\tilde{\rho}(g').\end{aligned}$$

Therefore

$$\begin{aligned}&\|\tilde{\rho}_1(gg') - \tilde{\rho}_1(g)\tilde{\rho}_1(g')\|\\ &\quad = \|\tilde{\rho}_1(gg')^{-1}\tilde{\rho}_1(g)\tilde{\rho}_1(g') - \mathrm{Id}\|\\ &\quad = \|\exp(\alpha(g, g'))\exp(-b(gg')\exp(b(g))\exp(\rho(g)b(g'))) - \mathrm{Id}\|\\ &\quad = \|\exp(\alpha(g, g') - b(gg') + b(g) + \rho(g)b(g')) - \mathrm{Id} + \delta\|\end{aligned}$$

where $\|\delta\| \leqq 3\varepsilon^2$. So $\|\tilde{\rho}_1(gg') - \tilde{\rho}_1(g)\tilde{\rho}_1(g')\| \leqq 5\varepsilon^2$ for all $g, g' \in G$. Proposition 2 is proved.

We now can prove Theorem 1. Let $\tilde{\rho} : G \to U$ be an ε-representation of an amenable group G. Let ε_n, $n \geqq 0$ be a sequence defined by $\varepsilon_0 = \varepsilon$, $\varepsilon_n = 5\varepsilon_{n-1}^2$. We inductively define ε_n-representations $\tilde{\rho}_n$ of G by successive applications of Proposition 2. It is clear that $\varepsilon_n \leqq \varepsilon/3^n$ and that $\|\tilde{\rho}_n(g) - \tilde{\rho}_{n-1}(g)\| \leqq \varepsilon/3^n$ for $n > 1$ and any $g \in G$. Therefore the sequence $\tilde{\rho}_n(g) \in U$ is convergent for any $g \in G$. Define

$$\pi(g) \overset{\text{def}}{=} \lim_{n \to \infty} \tilde{\rho}_n(g).$$

It is clear that $\pi : G \to U$ is a representation of G and

$$\|\pi(g) - \tilde{\rho}(g)\| \leqq \sum_{n=1}^{\infty} \|\tilde{\rho}_n(g) - \tilde{\rho}_{n-1}(g)\| \leqq \varepsilon.$$

Theorem 1 is proved.

Let X be a Riemannian surface of genus 2 and Γ be the fundamental group of X. We will show that for any $\varepsilon > 0$ there exists a finite-dimensional ε-representation $\tilde{\rho} : \Gamma \to U(N)$ such that for any representation $\pi : \Gamma \to U(N)$ we have $\sup_{g \in \Gamma} \|\tilde{\rho}(g) - \pi(g)\| > 1/10$.

We start with the following observation. Fix any integer N and denote by $D \subset U(N)$ the set of $u \in U$ such that $\|u - \mathrm{Id}\| < 1$, and denote by φ the continuous function on D given by

$$\varphi(u) \overset{\text{def}}{=} \frac{1}{2\pi i} \operatorname{Tr} \ln u.$$

Define $D' = D \cap SU(N)$.

LEMMA 4. *$\varphi(u') \in Z$ for any $u' \in D'$.*

PROOF. As is well known $\exp(2\pi i \varphi(u)) = \det u$ for any $u \in D$. Therefore $\exp(2\pi i \varphi(u')) = 1$ for any $u' \in D'$. The lemma is proved.

COROLLARY. *Define $D'_0 = \varphi^{-1}(0) \subset D'$, $\underline{D}' = D' - D'_0$. Then D'_0 and $\underline{D}'$ are open and closed subsets of D'.*

REMARK. It is easy to see that D'_0 is the connected component of Id in D'.

Let u, v be the elements of $U(N)$ and $w = uvu^{-1}v^{-1}$.

LEMMA 5. *If $\|w - \mathrm{Id}\| < 1/5$ and $w \in \underline{D}'$ then for any $u', v' \in U(N)$ such that $\|u - u'\|, \|v - v'\| < 1/5$, $u'v' \neq v'u'$.*

PROOF. Define $u(t)$, $v(t) \in U$, $0 \leqq t \leqq 1$ by

$$u(t) \stackrel{\text{def}}{=} u \exp(t \ln(u^{-1}u')), \qquad v(t) = v \exp(t \ln(v^{-1}v')).$$

It is clear that $u(t)$, $v(t)$ are continuous maps from $[0,1]$ to $U(N)$, $u(0) = u$, $u(1) = u'$, $v(0) = v$, $v(1) = v'$ and $\|u(t) - u\|$, $\|v(t) - v\| < 1/5$ for $0 \leqq t \leqq 1$. Define $w(t) = u(t)v(t)u(t)^{-1}v(t)^{-1}$. It is clear that $w(t)$ is a continuous map from $[0,1]$ to D'. Since $w(0) = w \in \underset{\sim}{D}'$ we have $w(1) = u'v'u'^{-1}v'^{-1} \in \underset{\sim}{D}'$. The lemma is proved.

We will need a slight generalization of this result. Let u_i, $1 \leqq i \leqq 4$ be elements of $U(N)$ and $w = u_1u_2u_1^{-1}u_2^{-1}u_3u_4u_3^{-1}u_4^{-1}$. Assume that $\|w - \mathrm{Id}\| < 1/10$ and $w \in \underset{\sim}{D}'$.

LEMMA 5'. *For any $u'_i \in U(N)$, $1 \leqq i \leqq 4$ such that $\|u_i - u'_i\| < 1/10$, $1 \leqq i \leqq 4$ we have $u'_1u'_2u_1'^{-1}u_2'^{-1}u'_3u'_4u_3'^{-1}u_4'^{-1} \neq \mathrm{Id}$.*

PROOF. The same.

Now consider our group Γ. It can be realised as a torsion-free co-compact subgroup of $\mathrm{SL}(2, R)$. Let S be the upper half plane which we consider as a Lobachevsky plane. $\mathrm{SL}(2, R)$ naturally acts on S and Γ is the fundamental group of $X = \Gamma \backslash S$. Let $\omega \in \Omega^2(S)$ be the $\mathrm{SL}(2, R)$-invariant differential form on S such that $\int_X \omega = 1$. As ω is Γ-invariant we can consider it as a two-form on X. Then ω represents a generator α of $H^2(X, R)$. By standard arguments we can identify $H^2(X, R)$ with $H^2(\Gamma, R)$. Fix $s_0 \in S$ and for any $g, g' \in \Gamma$ we denote by $c(g, g')$ the oriented area of the triangle with vertices $(s_0, gs_0, g's_0)$ in respect to ω. The following result is well known.

LEMMA 6. *$c(g, g')$ is a 2-co-cycle on Γ which represents α.*

The cohomology class $\alpha \in H^2(\Gamma, R)$ corresponds to a central extension of Γ

$$0 \longrightarrow R \longrightarrow \tilde{\Gamma}_R \xrightarrow{\pi} \Gamma \longrightarrow 0$$

and there exists a map $\delta : \Gamma \to \tilde{\Gamma}_R$ such that $\pi \circ \delta = \mathrm{Id}$ and $\delta(gg') = c(g, g') \cdot \delta(g)\delta(g')$ where we identify R with a subgroup in $\tilde{\Gamma}_R$.

Since $\int_X \omega = 1$ our class α lies in the image of $H^2(\Gamma, Z)$. Therefore, there exists a subgroup $\tilde{\Gamma} \subset \tilde{\Gamma}_R$ such that $\pi(\tilde{\Gamma}) = \Gamma$ and $\tilde{\Gamma} \cap R = Z$. For any $\tilde{\gamma} \in \tilde{\Gamma}_R$ we denote by $[\tilde{\gamma}]$ the unique element in $\tilde{\Gamma}$ such that $\tilde{\gamma} = a \cdot [\tilde{\gamma}]$ where $a \in R$, $0 \leqq a < 1$. We denote by δ' the map $\delta' : \Gamma \to \tilde{\Gamma}$ given by $\delta'(g) = [\delta(g)]$, $g \in \Gamma$ and define $c'(g, g')$ by $\delta'(gg') = c'(g, g')\delta'(g)\delta'(g')$ for $g, g' \in \Gamma$.

LEMMA 7. *$c'(g,g')\in Z$ and $|c'(g,g')|\leqq 3$ for $g,g'\in\Gamma$.*

PROOF. The first statement follows from the equality $R\cap\tilde\Gamma=Z$. To prove the second one we observe that the area of any triangle in S is $\leqq 1/2$ and therefore $|c(g,g')|\leqq 1/2$. The statement of the lemma now immediately follows from the definition of the co-cycle c'.

THEOREM 2. *For any $\varepsilon>0$ there exists a finite dimensional ε-representation $\tilde\rho:\Gamma\to U(N)$ such that for any representation $\pi:\Gamma\to U(N)$ one has*

$$\sup_{g\in\Gamma}\|\tilde\rho(g)-\pi(g)\|\geqq 1/10.$$

PROOF. As is well known Γ is a group generated by four generators γ_1, γ_2, γ_3, γ_4 and one relation $\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}\gamma_3\gamma_4\gamma_3^{-1}\gamma_4^{-1}=e$. The central extension $\tilde\Gamma$ is generated by five generators $\tilde\gamma_1$, $\tilde\gamma_2$, $\tilde\gamma_3$, $\tilde\gamma_4$, θ and relations $\tilde\gamma_i\theta=\theta\tilde\gamma_i$, $1\leqq i\leqq 4$ and $\theta=\tilde\gamma_1\tilde\gamma_2\tilde\gamma_1^{-1}\tilde\gamma_2^{-1}\tilde\gamma_3\tilde\gamma_4\tilde\gamma_3^{-1}\tilde\gamma_4^{-1}$. The projection $\pi:\tilde\Gamma\to\Gamma$ maps $\tilde\gamma_i$ to γ_i, $1\leqq i\leqq 4$ and $\pi(\theta)=e$. We will identify θ with the generators of the center Z in $\tilde\Gamma$.

Assume that $\varepsilon<1/10$ and fix N such that $N>3/\varepsilon$. Let $\eta=\exp(2\pi i/N)$, $A\subset U(N)$ be the diagonal matrix with elements $a_{kk}=\eta^k$, $1\leqq k\leqq N$ and $B=(b_{ij})\subset U(N)$ be the matrix given by

$$b_{ij}=\begin{cases}1 & \text{if } i-j=1 \pmod N,\\ 0 & \text{otherwise.}\end{cases}$$

It is clear that $ABA^{-1}B^{-1}=\eta\,\mathrm{Id}$. Let $\sigma:\tilde\Gamma\to U(N)$ be the representation given on generators by $\sigma(\tilde\gamma_1)=A$, $\sigma(\tilde\gamma_2)=B$, $\sigma(\tilde\gamma_3)=\sigma(\tilde\gamma_4)=\mathrm{Id}$, $\sigma(\theta)=\eta\,\mathrm{Id}$.

It is clear that all relations are satisfied and therefore the representation $\sigma:\tilde\Gamma\to U(N)$ is well defined. We now define the map $\tilde\rho:\Gamma\to U(N)$ by $\tilde\rho=\sigma\circ\delta'$. Then

$$\tilde\rho(gg')=\sigma(\delta'(gg'))=\sigma(\theta^{c'(g,g')}\delta'(g)\cdot\delta'(g'))=\eta^{c'(g,g')}\tilde\rho(g)\tilde\rho(g').$$

Therefore $\|\tilde\rho(gg')-\tilde\rho(g)\tilde\rho(g)\tilde\rho(g')\|=|\eta^{c'(g,g')}-1|$. Since $|c'(g,g')|\leqq 3$ and $N>3/\varepsilon$ we see that $\tilde\rho$ is an ε-representation of Γ.

Now let $\pi:\Gamma\to U(N)$ be any representation. We apply Lemma 5′ to $u_i=^{\mathrm{def}}\sigma(\tilde\gamma_i)$, $u_i'=^{\mathrm{def}}\pi(\gamma_i)$, $1\leqq i\leqq 4$.

Then $w=\eta\,\mathrm{Id}$ and $\varphi(w)=1$. On the other hand $\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}\gamma_3\gamma_4\gamma_3^{-1}\gamma_4^{-1}=e$ and therefore $u_1'u_2'u_1'^{-1}u_2'^{-1}u_3'u_4'u_3'^{-1}u_4'^{-1}=\mathrm{Id}$. It now follows from Lemma 5′ that $\max_{1\leqq i\leqq 4}\|u_i-u_i'\|\geqq 1/10$. Theorem 2 is proved.

Reference

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